ON SOME PROPERTIES OF TWO VECTOR-VALUED VAR AND CTE MULTIVARIATE RISK MEASURES FOR ARCHIMEDEAN COPULAS

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ABSTRACT

We consider the multivariate Value-at-Risk (VaR) and Conditional-Tail-Expectation (CTE) risk measures introduced in Cousin and Di Bernardino (COUSIN, A. and DI BERNARDINO, E. (2013) *Journal of Multivariate Analysis*, **119**, 32–46; COUSIN, A. and DI BERNARDINO, E. (2014) *Insurance: Mathematics and Economics*, **55**(C), 272–282). For absolutely continuous Archimedean copulas, we derive integral formulas for the multivariate VaR and CTE Archimedean risk measures. We show that each component of the multivariate VaR and CTE functional vectors is an integral transform of the corresponding univariate VaR measures. For the class of Archimedean copulas, the marginal components of the CTE vector satisfy the following properties: positive homogeneity (PH), translation invariance (TI), monotonicity (MO), safety loading (SL) and VaR inequality (VIA). In case marginal risks satisfy the subadditivity (MSA) property, the marginal CTE components are also sub-additive and hitherto coherent risk measures in the usual sense. Moreover, the increasing risk (IR) or stop-loss order preserving property of the marginal CTE components holds for the class of bivariate Archimedean copulas. A counterexample to the (IR) property for the trivariate Clayton copula is included.

KEYWORDS

multivariate risk, coherent risk measure, increasing risk, Archimedean copula, Kendall's process.

1. INTRODUCTION

The univariate risk measures of *Value-at-Risk* (VaR) and *Conditional-Tail-Expectation* (CTE) are well-known mathematical notions used in quantitative risk measurement (e.g. Albrecht, 2004; McNeil *et al.*, 2005). In recent years, various efforts have been undertaken to extend these risk measures to the multivariate context. Extensions for *multivariate VaR* have been discussed by Tibiletti

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(1993), Embrechts and Puccetti (2006) and Nappo and Spizzichino (2009), and for *multivariate CTE* by Landsman and Valdez (2003), Hürlimann (2004a), Cai and Li (2005) and Bargès *et al.* (2009). Cousin and Di Bernardino (2013, 2014) have proposed multivariate VaR and CTE risk measures in a rather parsimonious and synthetic way by assigning to the risk of a multi-dimensional portfolio two vector valued functional risk measures. These authors have shown that many properties satisfied by the univariate VaR and CTE risk measures translate to the multivariate setting under some conditions. For example, these risk measures satisfy positive homogeneity (PH) and translation invariance (TI) properties, which are parts of the classical axiomatic approach to coherent risk measures by Artzner *et al.* (1999). In fact, Cousin and Di Bernardino (2014) contains all the properties required to imply that the marginal CTE components of multivariate risks with independent components are coherent risk measures, as observed in Section 5. In view of these promising results, we follow this most recent vector-valued approach to multivariate risk measures. To be specific, our contribution includes some new properties that are not discussed in previous papers. The marginal CTE components of multivariate risks under a fixed Archimedean copula satisfy the additional properties of safety loading (SL) and VaR inequalities (VIA) (see Theorem 5.2). The marginal VaR and CTE Archimedean risk measures are coherent risk measures, provided the marginal risk components are subadditive, the so-called (MSA) property. The increasing risk (IR) property or stop-loss order preserving property holds for the marginal CTE components of bivariate risks with an Archimedean copula (see Corollary 5.1). Main tools in the derivation of these results are two representations of the multivariate VaR and CTE Archimedean risk measures as integral transforms of the corresponding univariate VaR measures (see Theorem 4.2).

The content is organized as follows. Section 2 gathers the required preliminaries on copulas, Kendall's distribution and stochastic orders. Section 3 introduces the multivariate VaR and CTE functional vectors and recalls two general formulas for their evaluation.

Section 4 is devoted to a brief treatment of multivariate VaR and CTE for an important class of Archimedean copulas. Based on the explicit expressions of Kendall's distribution and its density function by Barbe *et al.* (1996), and a conditional distribution formula for absolutely continuous Archimedean copulas, we derive integral representation formulas for the considered multivariate risk measures. In particular, we show that each component of the multivariate VaR or CTE vector is an integral transform of the associated univariate VaR measure.

Section 5 reviews some main properties of multivariate CTE (Theorem 5.1), and derives new properties for the class of Archimedean copulas. Altogether, the marginal CTE components for this class satisfy the following properties: (PH), (TI), monotonicity (MO), (SL) and (VIA) (Theorem 5.2). In case marginal risks satisfy the (MSA) property, the marginal CTE components are also subadditive and hitherto coherent risk measures. Moreover, the (IR) or stop-loss order preserving property of the marginal CTE components holds for the class of bivariate Archimedean copulas (Corollary 5.1). A counterexample to the (IR) property for the trivariate Clayton copula is constructed in Example 5.2. In general, examples involving Archimedean copulas can be derived based on the material of Barbe *et al.* (1996), Nelsen (2006) and Cousin and Di Bernardino (2013, 2014), among others.

2. Preliminaries on copulas, Kendall's distribution and stochastic orders

Roughly speaking, a d-dimensional *copula* $C_d(u)$, $u = (u_1, \dots, u_d) \in [0, 1]^d$, is a d-dimensional distribution function with uniform [0,1] univariate margins. The Theorem of Sklar (1959) establishes the following fundamental link between an d-dimensional distribution function $F(x)$, $x = (x_1, \dots, x_d) \in R^d$ and its univariate marginal distribution functions $F_i(x_i)$, $i = 1, \dots, d$. For continuous margins there exists a uniquely determined copula $C_d(u)$ such that

$$F(x) = C_d(F_1(x_1), \dots, F_d(x_d)), x \in R^d. \tag{2.1}$$

Each copula is the sum of an *absolutely continuous component* $A_C(u)$ and a *singular component* $S_C(u)$ such that $C_d(u) = A_C(u) + S_C(u)$, where these components are defined by

$$A_C(u) = \int_{(0,u]} c_d(x)dx, \; S_C(u) = A_C(u) - C_d(u), \tag{2.2}$$

with $c_d(u)$ being the Radon–Nikodym derivative with respect to the Lebesgue measure on $[0, 1]^d$. If $C_d(u) = A_C(u)$, then the copula is *absolutely continuous*, whereas if $C_d(u) = S_C(u)$, the copula is *singular*. In the present work, we mainly focus on absolutely continuous Archimedean copulas whose density function is explicitly known (see Theorem 4.1).

Consider now a d-dimensional random vector $X = (X_1, \dots, X_d)$ on some probability space with multivariate distribution $F(x)$. The multivariate risk measures introduced in Section 3 strongly rely on the key notion of multivariate probability integral transform, or *multivariate PIT*, defined to be the univariate random variable $Z = F(X)$, whose corresponding distribution function, which is defined and denoted by $K_d(z) = P(Z \leq z)$, is called the *Kendall distribution*. The survival distribution function of $Z = F(X)$ is also used. It is defined and denoted by $\bar{K}_d(z) = P(Z > z)$. In contrast to the univariate case, the distribution $K_d(z)$, $d \geq 2$, is in general not uniform on [0,1], even if $F(x)$ is continuous. The random vector $U = (U_1, \dots, U_d)$ denotes throughout a vector of uniform [0,1] random variables. As a consequence of Sklar's theorem, the Kendall distribution is a function of the dependence structure only, that is, on the copula $C_d(u)$ that belongs to X. In particular, one has $K_d(z) = P(C_d(U) \leq z)$ with $U = (F_1(X_1), \dots, F_d(X_d))$. More information on the multivariate PIT, Kendall's distribution and its applications is found in the papers by Genest and Rivest (1993, 2001), Capéraà *et al.* (1997), Ghoudi *et al.* (1998), Chakak and Ezzerg (2000), Chakak and Imlahi (2001), Nelsen *et al.* (2001, 2003), Genest and Boies (2003), Genest *et al.* (2002, 2006), Kolev *et al.* (2006), Belzunce *et al.* (2007) and Brechmann (2014), among others. Kendall's distribution and density function for Archimedean copulas is summarized in (4.2) and (4.3) in Section 4.

Finally, to study the comparison properties of these risk measures in Section 5, we rely on the theory of stochastic orders (e.g. Kaas *et al.*, 1994; Shaked and Shanthikumar, 1994, 2007; Müller and Stoyan, 2002; Denuit and Müller, 2004; Belzunce, 2010). Let X and Y be two univariate random variables defined on some probability space with distribution functions $F_X(x)$, $F_Y(x)$, and finite means μ_X, μ_Y. Then X is said to be smaller than Y in the usual *stochastic order*, written as $X \leq_{st} Y$ if for all $x \in R$ the inequality $F_X(x) \geq F_Y(x)$ holds. On the other hand, X is said to be smaller than Y in the *stop-loss order*, written as $X \leq_{sl} Y$ if for all $t \in R$ the stop-loss transform inequality $E[(X-t)_+] \leq E[(Y-t)_+]$ holds with $x_+ = \max(x, 0)$. Similarly, X is said to be smaller than Y in *increasing convex order*, written as $X \leq_{icx} Y$, if $E[f(X)] \leq E[(f(Y)]$ for all non-decreasing convex functions $f(x)$ such that the expectations exist. The stop-loss order and the increasing convex order are equivalent (e.g. Shaked and Shanthikumar, 2007, Theorem 4.A.2). A sufficient condition for the stop-loss order is the *dangerousness order* relation, written as $X \leq_D Y$, which is defined by the once-crossing condition,

$$F_X(c) \leq F_Y(c) \forall x < c, \quad F_X(c) \geq F_Y(c) \forall x \geq c, \tag{2.3}$$

where c is some real number, and the inequality $\mu_X \leq \mu_Y$ (Karlin and Novikoff, 1963; Ohlin, 1969, Lemma; see also Hürlimann, 1998, Section 2).

3. Two vector-valued multivariate VaR and CTE formulas

The univariate risk measures of VaR and CTE are well-known mathematical notions used in the context of risk management. A *univariate risk* is a non-negative random variable X on a given probability space with a continuous and strictly monotone distribution function $F_X(x)$ with finite mean $E[X] < \infty$. Its quantile function is denoted by $Q_X(\alpha) = F_X^{-1}(\alpha), \alpha \in (0, 1)$. The VaR functional to the confidence level α is defined and denoted by

$$VaR_\alpha[X] = Q_X(\alpha), \alpha \in (0, 1). \tag{3.1}$$

Similarly, the CTE functional to the confidence level α is defined and denoted by

$$CTE_\alpha[X] = E[X | X \geq VaR_\alpha[X]], \alpha \in (0, 1). \tag{3.2}$$

For a comprehensive treatment of VaR and CTE we refer to Denuit *et al.* (2005). A list of many different possible representations for the CTE functional is given in Hürlimann (2003). A first extensive review of CTE is Nadarajah *et al.* (2013).

In the following, a *multivariate risk* is a non-negative absolutely continuous random vector $X = (X_1, \ldots, X_d)$ (w.r.t. the Lebesgue measure on R^d) with a partially increasing multivariate distribution function $F : R_+^d \to [0, 1]$ with finite marginal means $E[X_i] < \infty, i = 1, \ldots, d$. One says that the random vector X satisfies the so-called *regularity conditions*. For any $\alpha \in (0, 1)$, let $L(\alpha) = \left\{x \in R_+^d : F(x) \geq \alpha\right\}$ be the *upper α-level set* of F, and let $\partial L(\alpha)$ be its boundary. Then the *multivariate VaR* functional to the confidence level α is the d-dimensional vector defined and denoted by Cousin and Di Bernardino (2013, Definition 2.1, lower-orthant VaR),

$$\begin{aligned} VaR_\alpha[X] = \left(VaR_\alpha^1[X], \ldots, VaR_\alpha^d[X]\right), \; VaR_\alpha^i[X] \\ = E[X_i \,|\, X \in \partial L(\alpha)], i = 1, \ldots, d. \end{aligned} \tag{3.3}$$

Similarly, the *multivariate CTE* functional to the confidence level α is the d-dimensional vector defined and denoted by Cousin and Di Bernardino (2014, Definition 2.1, lower-orthant CTE),

$$\begin{aligned} CTE_\alpha[X] = \left(CTE_\alpha^1[X], \ldots, CTE_\alpha^d[X]\right), \quad CTE_\alpha^i[X] \\ = E[X_i \,|\, X \in L(\alpha)], i = 1, \ldots, d. \end{aligned} \tag{3.4}$$

It is remarkable that each vector component of multivariate VaR can be represented as an integral transform of the associated univariate VaR measure. Indeed, for $i = 1, \ldots, d$, let F_{X_i} be the marginal distribution of X_i, and let $C_d(u)$ be the copula associated with X such that by Sklar's theorem (1959) we have $F(x_1, \ldots, x_d) = C_d(F_{X_1}(x_1), \ldots, F_{X_d}(x_d))$. Then the random variables $U_i = F_{X_i}(X_i)$ are uniform [0,1] random variables with joint distribution $C_d(u)$. From Cousin and Di Bernardino (2013, equation (12)), we borrow the general formula,

$$VaR_\alpha^i[X] = K_d'(\alpha)^{-1} \cdot \int_\alpha^1 VaR_s[X_i] \cdot f_d(s, \alpha) ds, \quad i = 1, \ldots, d, \tag{3.5}$$

where $f_d(x_i, z)$ is the density function associated with the bivariate random vector $(U_i, C_d(U)), i = 1, \ldots, d$, with $C_d(U)$ being the multivariate PIT, and $K_d(x)$ being the Kendall distribution, both introduced in Section 2. Moreover, each component of multivariate CTE can be represented as an integral transform of the corresponding multivariate VaR component. Cousin and Di Bernardino (2014, equation (8)), derive the following general formula:

$$CTE_\alpha^i[X] = \bar{K}_d(\alpha)^{-1} \cdot \int_\alpha^1 VaR_s^i[X] \cdot K_d'(s) ds, \quad i = 1, \ldots, d. \tag{3.6}$$

4. The Archimedean copula framework

According to Nelsen (2006, Definition 4.1.1), an *Archimedean generator*, or for short a *generator*, is a strictly decreasing and continuous function $\varphi : [0, 1] \to [0, \infty]$ such that $\varphi(1) = 0$. Its pseudo-inverse is denoted by φ^- (notation of Embrechts and Hofert, 2013). Let Φ denote the set of all generators. A d-dimensional copula C_d is called *Archimedean* if it can be represented as (Nelsen, 2006, equation (4.6.1))

$$C_d(u) = \varphi^-(\varphi(u_1) + ... + \varphi(u_d)), u \in [0, 1]^d, \tag{4.1}$$

for some $\varphi \in \Phi$. Early contributions to Archimedean copulas include Schweizer and Sklar (1961) and Genest and MacKay (1986a). In higher dimensions, it is often more appropriate to work directly with the pseudo-inverse φ^- instead of φ (e.g. Joe, 1997; McNeil and Neslehova, 2009). If φ is a *strict* generator, that is $\varphi(0) = \infty$, one knows from the Kimberling (1974) theorem that (4.1) is well defined if and only if $\varphi^- = \varphi^{-1}$ is completely monotone on $[0, \infty)$, that is $(-1)^k(\varphi^{-1})^{(k)}(x) > 0, k = 0, 1, 2, \ldots, x \in (0, \infty)$ (see Nelsen, 2006, Theorem 4.6.2). In general, (4.1) is well defined if and only if φ^{-1} is d-monotone on $[0, \infty)$, as shown by McNeil and Neslehova (2009, Theorem 2) (see also Nelsen, 2006, p. 154). For simplicity, we assume throughout that φ^{-1} is completely monotone.

We require Kendall's distribution and its density function for Archimedean copulas, which have been determined in Barbe *et al.* (1996) (see also Genest and Rivest, 2001, Section 5).

Lemma 4.1. (*Kendall's distribution for Archimedean copulas*) *Under the technical assumptions* $(-1)^k(\varphi^{-1})^{(k)}(x) > 0, k = 1, \ldots, d$, *and* $\lim_{x\to 0^+} \varphi(x)^k(\varphi^{-1})^{(k)}(\varphi(x)) = 0, k = 1, \ldots, d-1$, *one has*

$$K_d(z) = K_{d-1}(z) + \frac{(-1)^{d-1}}{(d-1)!}\varphi(z)^{d-1}(\varphi^{-1})^{(d-1)}(\varphi(z)), \tag{4.2}$$

$$K_d'(z) = \frac{(-1)^{d-1}}{(d-1)!}\varphi(z)^{d-1}\frac{d}{dz}(\varphi^{-1})^{(d-1)}(\varphi(z)). \tag{4.3}$$

Proof. Define recursively $f_0(z) = 1/\varphi'(z)$, $f_k(z) = \frac{d}{dz} f_{k-1}(z)/\varphi'(z)$, $k = 1, 2, \ldots, d-1$, to see that

$$f_{k-1}(z) = (\varphi^{-1})^{(k)}(\varphi(z)), \quad k = 1, \ldots, d.$$

From Barbe *et al.* (1996, Example 3, p. 205), one knows that

$$K_d(z) = z + \sum_{k=1}^{d-1} \tfrac{(-1)^k}{k!}\varphi(z)^k f_{k-1}(z).$$

Using the preceding relation, induction and the fact that $K_1(z) = z$, one sees that this finite series is equivalent with Formula (4.2). Taking derivatives one obtains

$$\begin{aligned}
&K_d'(z)\\
&= 1 + \sum_{k=1}^{d-1} \frac{(-1)^k}{k!}\{k\varphi(z)^{k-1}\varphi'(z)f_{k-1}(z) + \varphi(z)^k\varphi'(z)f_k(z)\}\\
&= 1 - \varphi'(z)f_0(z) - \sum_{j=1}^{d-2}\frac{(-1)^j}{j!}\varphi(z)^j\varphi'(z)f_j(z) + \sum_{k=1}^{d-1}\frac{(-1)^k}{k!}\varphi(z)^k\varphi'(z)f_k(z)\\
&= \frac{(-1)^{d-1}}{(d-1)!}\varphi(z)^{d-1}\varphi'(z)(\varphi^{-1})^{(d)}(\varphi(z)) = \frac{(-1)^{d-1}}{(d-1)!}\varphi(z)^{d-1}\frac{d}{dz}(\varphi^{-1})^{(d-1)}(\varphi(z)),
\end{aligned}$$

which is Formula (4.3). □

Multivariate Archimedean copulas, for which Lemma 4.1 applies, include copula families by Clayton, Frank, Gumbel–Hougaard and Ali–Mikhail–Haq (see Barbe *et al.*, 1996, pp. 206–207).

According to (3.5)–(3.6), the multivariate VaR and CTE depend upon the bivariate random vector $(U_i, C_d(U))$, $i = 1, \ldots, d$, with distribution function $F_d(x_i, z) = P(U_i \leq x_i, C_d(U) \leq z)$ and density $f_d(x_i, z)$. The latter has the following closed-form expression.

Theorem 4.1. *Let C_d be an absolutely continuous d-dimensional Archimedean copula with generator φ. Then, the density function $f_d(x_i, z)$ is given by*

$$\begin{aligned}
&f_d(x_i, z) = -\frac{(d-1)\cdot\varphi'(x_i)}{\varphi(z)}\cdot\left(1 - \frac{\varphi(x_i)}{\varphi(z)}\right)^{d-2}\cdot K_d'(z), \quad 0 < z < x_i < 1,\\
&f_d(x_i, z) = 0, \quad 0 \leq x_i \leq z \leq 1.
\end{aligned} \tag{4.4}$$

Proof. First of all, since a copula is non-decreasing in each argument, the condition $U_i \leq x_i \leq z$ implies that $C_d(U) \leq C_d(U_1, \ldots, U_{i-1}, z, U_{i+1}, \ldots, U_d) \leq C_d(1, \ldots, 1, z, 1, \ldots, 1) = z$. Therefore, if $0 \leq x_i \leq z \leq 1$, one has $F_d(x_i, z) = P(U_i \leq x_i, C_d(U) \leq z) = P(U_i \leq x_i) = x_i$. Under absolute continuity this implies in particular that $f_d(x_i, z) = \partial^2 F_d(x_i, z)/\partial x_i \partial z = 0$, $0 \leq x_i \leq z \leq 1$, which shows the second part of Formula (4.4). On the other hand, for $0 < z < x_i < 1$, the conditional distribution of U_i, given $C_d(U) = z$, is of the simple form (e.g. Cousin and Di Bernardino, 2013, Formula (8); or Brechmann, 2014, Lemma

17, special case $j = 1$)

$$F_d(x_i\,|z) = P(U_i \le x_i\,|C_d(U) = z) = \left(1 - \frac{\varphi(x_i)}{\varphi(z)}\right)^{d-1}. \tag{4.5}$$

The first part of (4.4) follows from the fact that $f_d(x_i, z) = \frac{d}{dx_i} F_d(x_i\,|z) \cdot f_{C_d(U)}(z)$ with $f_{C_d(U)}(z) = K'(z)$. □

Remark 4.1. For a density of the form (4.4), the assumption of absolute continuity is required. Consider a twice differentiable generator such that $\varphi'(x) < 0, \varphi''(x) > 0$ for all $x \in (0, 1)$. Then in the bivariate case with $x_i = x$, a density of the type $f_2(x, z) = -\varphi'(x) \cdot \varphi''(z)/\varphi'(z)^2$ for $0 < z < x < 1$ yields through integration the distribution function

$$F_2(x, z) = \int_0^z \left(\int_y^x f_2(s, t) ds\right) dt = K_2(z) - K_2(0) + \varphi(x) \cdot \left(\varphi'(z)^{-1} - \varphi'(0)\right),$$
$$K_2(z) = z - \varphi(z) \cdot \varphi'(z)^{-1}, \quad K_2(0) = -\varphi(0) \cdot \varphi'(0)^{-1}. \tag{4.6}$$

It is known that whether the bivariate Archimedean copula is absolutely continuous or not, one always has (Nelsen, 2006, Corollaries 4.3.5 and 4.3.6),

$$\begin{aligned} &F_2(x, z) = K_2(z) + \varphi(x) \cdot \varphi'(z)^{-1}, \quad 0 < z < x < 1, \\ &F_d(x, z) = x, 0 \le x \le z \le 1. \end{aligned} \tag{4.7}$$

Through the comparison of (4.6) with (4.7) one must necessarily have $K_2(0) = 0$ and $\varphi'(0) = -\infty$. If φ is a *strict* generator, that is $\varphi(0) = \infty$, then $K_2(0) = 0$ implies $\varphi'(0) = -\infty$. If φ is a *non-strict* generator, that is $\varphi(0) < \infty$, then $K_2(0) = 0$ if and only if one has $\varphi'(0) = -\infty$. It follows that (4.6) and (4.7) are equal if and only if $K_2(0) = 0$, which is a necessary and sufficient condition for absolute continuity. Indeed, for twice differentiable generators an Archimedean copula has a singular component if and only if one has $K_2(0) = -\varphi(0) \cdot \varphi'(0)^{-1} \neq 0$ (Genest and MacKay, 1986b, Theorem 1). An Archimedean copula with singular component for which Formulas (4.6) and (4.7) differ is the family (4.2.2) in Nelsen (2006, Table 4.1) with non-strict generator $\varphi(x) = (1 - x)^{\theta}, \theta \in [1, \infty)$, such that $K_2(0) = 1/\theta > 0, \varphi'(0) = -\theta > -\infty$.

We use Theorem 4.1 to determine the multivariate VaR functional for Archimedean copulas. Moreover, we show that each component of the multivariate CTE functional, similar to its VaR pendant, is also an integral transform of the associated univariate VaR measure.

Theorem 4.2. *Let C_d be an absolutely continuous d-dimensional Archimedean copula with generator φ. Then the components $i = 1, \ldots, d$ of the multivariate VaR* *and CTE functional satisfy the following formulas:*

$$VaR_\alpha^i[X] = \frac{d-1}{\varphi(\alpha)^{d-1}} \cdot \int_\alpha^1 VaR_s[X_i] \cdot \beta_d(s,\alpha) ds, \quad \beta_d(s,\alpha) = -\varphi'(s) \cdot (\varphi(\alpha) - \varphi(s))^{d-2}, \tag{4.8}$$

$$\begin{aligned} CTE_\alpha^i[X] &= A_d(\alpha) \cdot \int_\alpha^1 VaR_s[X_i] \cdot \gamma_d(s,\alpha) ds, \quad i = 1, \ldots, d, \\ A_d(\alpha) &= [(d-2)!]^{-1} \cdot \bar{K}_d(\alpha)^{-1}, \quad \gamma_d(s,\alpha) \\ &= \int_\alpha^s (-1)^{d-1} \frac{d}{dt} (\varphi^{-1})^{(d-1)}(\varphi(t)) \cdot \beta_d(s,t) dt. \end{aligned} \tag{4.9}$$

Proof. To derive (4.8), it suffices to insert (4.4) into (3.5). To derive (4.9) we use (4.8) and exchange the order of integration in representation (3.6) (using Formula (4.3) and Fubini's theorem) to see that

$$\begin{aligned} CTE_\alpha^i[X] &= \bar{K}_d(\alpha)^{-1} \cdot \int_\alpha^1 VaR_s^i[X] \cdot K_d'(s) ds \\ &= (d-1) \cdot \bar{K}_d(\alpha)^{-1} \cdot \int_\alpha^1 \left\{ \int_s^1 VaR_t[X_i] \cdot \beta_d(t,s) dt \right\} \cdot K_d'(s)/\varphi(s)^{d-1} ds \\ &= A_d(\alpha) \cdot \int_\alpha^1 \left\{ \int_s^1 VaR_t[X_i] \cdot \beta_d(t,s) dt \right\} \cdot (-1)^{d-1} \frac{d}{ds} (\varphi^{-1})^{(d-1)}(\varphi(s)) ds \\ &= A_d(\alpha) \cdot \int_\alpha^1 \left\{ \int_\alpha^t (-1)^{d-1} \frac{d}{ds} (\varphi^{-1})^{(d-1)}(\varphi(s)) \cdot \beta_d(t,s) ds \right\} \cdot VaR_t[X_i] dt, \end{aligned}$$

which coincides with (4.9) by the definition of $\gamma_d(s,\alpha)$. □

Remark 4.2. If for $d = 1$ one sets $A_1(\alpha) = \bar{K}_1(\alpha)^{-1} = (1-\alpha)^{-1}, \gamma_1(s,\alpha) = 1, \forall s \in [\alpha, 1]$, then Formula (4.9) generalizes the well-known univariate formula $CTE_\alpha[X_1] = (1-\alpha)^{-1} \cdot \int_\alpha^1 VaR_s[X_1] ds$ (e.g. Hürlimann, 2003, Proposition 2.1).

As an application of Formula (4.8), we show that the multivariate VaR components are non-decreasing functions of the confidence level, a result used in the proof of Theorem 5.2.

Corollary 4.1. *Let C_d be an absolutely continuous d-dimensional Archimedean copula with generator φ. Then the derivative with respect to the confidence level of* *the multivariate VaR functional is non-negative, that is* $dVaR_\alpha^i[X]/d\alpha \geq 0, i = 1, \ldots, d, \forall\alpha \in (0, 1)$.

Proof. We distinguish between the two cases $d = 2$ and $d \geq 3$. If $d = 2$, one obtains from (4.8) through straightforward calculation that

$$\frac{dVaR_\alpha^i[X]}{d\alpha} = -\frac{\varphi'(\alpha)}{\varphi(\alpha)} \cdot \left\{ VaR_\alpha^i[X] - VaR_\alpha[X_i] \right\}.$$

The derivative is non-negative because the generator φ is strictly decreasing and $VaR_\alpha^i[X] - VaR_\alpha[X_i] \geq 0$ for bivariate Archimedean copulas. In fact, the latter inequality holds true more generally for quasi-concave distribution functions defined to be distribution functions with convex upper level sets (Cousin and Di Bernardino, 2013, Proposition 2.4). Its validity for bivariate Archimedean copulas follows from Nelsen (2006, Theorem 4.3.2). If $d \geq 3$, one obtains from (4.8) using Leibniz' rule of differentiation under the integral sign that

$$\frac{dVaR_\alpha^i[X]}{d\alpha} = -\frac{(d-1)\varphi'(\alpha)}{\varphi(\alpha)^d} \cdot \int_\alpha^1 VaR_s^i[X] \cdot (-\varphi'(s))(\varphi(\alpha) - \varphi(s))^{d-3}\{\varphi(\alpha) - \varphi(s) - (d-2)\varphi'(\alpha)\}ds,$$

which is non-negative because φ is strictly decreasing and $\varphi(\alpha) - \varphi(s) \geq 0 \forall s \in [\alpha, 1]$. □

To conclude this section, let us illustrate Formula (4.9) for some lower-dimensional cases.

Examples 4.1. Bivariate and trivariate CTE formulas.

For $d = 2$ one has $\gamma_2(s, \alpha) = -\varphi'(s) \cdot \int_\alpha^s \frac{d}{dt}(\varphi^{-1})'(\varphi(t))dt = -\varphi'(s) \cdot \{\varphi'(\alpha)^{-1} - \varphi'(s)^{-1}\}$. It follows that for $i = 1, 2$,

$$CTE_\alpha^i[X] = \bar{K}_2(\alpha)^{-1} \cdot \int_\alpha^1 VaR_s[X_i] \cdot \left\{1 - \frac{\varphi'(s)}{\varphi'(\alpha)}\right\} ds. \tag{4.10}$$

In the special case of uniform margins, one easily gets

$$CTE_\alpha^i[X] = \bar{K}_2(\alpha)^{-1} \cdot \left\{ \tfrac{1}{2}(1-\alpha^2) + \tfrac{\alpha\varphi(\alpha) + \int_\alpha^1 \varphi(s)ds}{\varphi'(\alpha)} \right\}. \tag{4.11}$$

For the comprehensive Clayton copula family $\varphi(s) = \theta^{-1}(s^{-\theta} - 1), \theta \geq -1$, which includes the independent copula ($\theta \to 0$), the co-monotone copula ($\theta \to \infty$) and the countermonotone copula ($\theta = -1$), one recovers Table 6 in Cousin and Di Bernardino (2014) (use that $K_2(\alpha) = \alpha - \varphi(\alpha)/\varphi'(\alpha)$). For

$d = 3$, Formula (4.9) for $i = 1, 2, 3$ yields

$$CTE_\alpha^i[X] = \bar{K}_3(\alpha)^{-1} \cdot \int_\alpha^1 VaR_s[X_i] \cdot \left\{ 1 - \frac{\varphi'(s)}{\varphi'(\alpha)} - \frac{\varphi''(\alpha)}{\varphi'(\alpha)^3} \varphi'(s)(\varphi(\alpha) - \varphi(s)) \right\} ds. \tag{4.12}$$

For uniform margins, one obtains the formula

$$CTE_\alpha^i[X] = \bar{K}_3(\alpha)^{-1} \cdot \left\{ \tfrac{1}{2}(1 - \alpha^2) - \int_\alpha^1 \left\{ \frac{s\varphi'(s)}{\varphi'(\alpha)} - \frac{\varphi''(\alpha)}{\varphi'(\alpha)^3} s\varphi'(s)(\varphi(\alpha) - \varphi(s)) \right\} ds \right\}. \tag{4.13}$$

5. Marginal properties of the vector-valued multivariate CTE functional

Some main properties of multivariate VaR and CTE are derived in Cousin and Di Bernardino (2013, 2014). In particular, for multivariate risks with independent components, the marginal CTE components are consistent with the desirable properties of a coherent risk measure. Here we ask for additional conditions under which the marginal VaR and CTE components are coherent risk measures, at least for Archimedean copulas. In the more restricted Archimedean copula framework, we investigate also conditions under which the marginal CTE components preserve the more stringent stop-loss order instead of the usual stochastic order (or the (MO) property as stated below). We begin with some properties, which hold without restriction.

Theorem 5.1. (*General marginal properties of multivariate CTE*) *Let $X = (X_1, \ldots, X_d)$ and $Y = (Y_1, \ldots, Y_d)$ be multivariate risks that satisfy the regularity conditions. Assume that X and Y belong to a fixed absolutely continuous copula $C_d(u)$ with density $c_d(u)$. Then the following properties are fulfilled:*

(PH) Positive Homogeneity: $\forall(\lambda_1, \ldots, \lambda_d) \in R_+^d$

$$CTE_\alpha^i[\lambda_i \cdot X] = \lambda_i \cdot CTE_\alpha^i[X], i = 1, \ldots, d, \quad \forall\alpha \in (0, 1). \tag{5.1}$$

(TI) Translation invariance: $\forall(\lambda_1, \ldots, \lambda_d) \in R_+^d$

$$CTE_\alpha^i[\lambda_i + X] = \lambda_i + CTE_\alpha^i[X], \quad i = 1, \ldots, d, \quad \forall\alpha \in (0, 1). \tag{5.2}$$

(SL) Safety Loading: If $VaR_\alpha^i[X] \leq VaR_\beta^i[X], \quad i = 1, \ldots, d, \quad \forall 0 < \alpha \leq \beta < 1$, then

$$CTE_\alpha^i[X] \geq E[X_i], \quad i = 1, \ldots, d, \quad \forall \alpha \in (0, 1). \tag{5.3}$$

(VIA) VaR Inequality: If $VaR_\alpha^i[X] \leq VaR_\beta^i[X], \quad i = 1, \ldots, d, \forall 0 < \alpha \leq \beta < 1$, then

$$CTE_\alpha^i[X] \geq VaR_\alpha^i[X], i = 1, \ldots, d, \forall \alpha \in (0, 1). \tag{5.4}$$

(MO) Monotonicity: If $X_i \leq_{st} Y_i, i = 1, \ldots, d$, then

$$CTE_\alpha^i[X] \leq CTE_\alpha^i[Y], i = 1, \ldots, d, \forall \alpha \in (0, 1). \tag{5.5}$$

Proof. Properties (PH) and (TI) are shown in Cousin and Di Bernardino (2014, Proposition 2.2). Properties (SL), (VIA) and (MO) are shown in Cousin and Di Bernardino (2014, Corollary 2.5, Proposition 2.8 and Proposition 2.10). □

If the fixed copula is an Archimedean copula, then (5.3) and (5.4) hold without assumption and the multivariate VaR and CTE are always greater or equal to univariate VaR.

Theorem 5.2. (*General marginal properties of multivariate CTE for Archimedean copulas*) *Let $X = (X_1, \ldots, X_d)$ and $Y = (Y_1, \ldots, Y_d)$ be multivariate risks that belong to an absolutely continuous d-dimensional Archimedean copula with generator φ. Then, besides (PH), (TI) and (MO), shown in Theorem 5.1, the following properties are fulfilled:*

(SL) Safety Loading:

$$CTE_\alpha^i[X] \geq E[X_i], \quad i = 1, \ldots, d, \quad \forall \alpha \in (0, 1). \tag{5.6}$$

(VIA) VaR Inequalities

$$CTE_\alpha^i[X] \geq VaR_\alpha^i[X] \geq VaR_\alpha[X_i], \quad i = 1, \ldots, d, \quad \forall \alpha \in (0, 1). \tag{5.7}$$

Proof. The proof that the two properties (SL) and (VIA) hold automatically for Archimedean copulas follows from Corollary 4.1, which states that the multivariate VaR components are non-decreasing functions of the confidence level. Finally, as observed in the proof for $d = 2$ of Corollary 4.1, the second inequality in (VIA) holds true in general for quasi-concave distribution functions, defined to be distribution functions with convex upper level sets (Cousin and Di Bernardino, 2013, Proposition 2.3). In particular, this holds for Archimedean copulas (Nelsen, 2006, Theorem 4.3.2, for $d = 2$, and Tibiletti, 1995, for arbitrary d). □

Next, we ask whether the marginal CTE components are coherent risk measures. This is the case, provided (PH), (TI), (MO) in Theorem 5.1 and the following subadditivity property holds:

(SA) CTE Subadditivity:

$$CTE^i_\alpha[X+Y] \leq CTE^i_\alpha[X] + CTE^i_\alpha[Y], \quad i = 1, \ldots, d, \quad \forall \alpha \in (0,1). \quad (5.8)$$

Cousin and Di Bernardino (2014, Proposition 2.4), shows that (SA) is fulfilled, provided the components of the multivariate risks are independent. In this situation, the marginal CTE components are coherent risk measures. On the other hand, one notes that the (SA) property does not hold for arbitrary margins (e.g. Lee and Prékopa, 2013, for a counterexample in the multivariate discrete case).

One can ask for additional conditions under which the marginal CTE components will be coherent risk measures. In general, the (SA) property of multivariate CTE for Archimedean copulas is inherited from the VaR subadditive property for the marginal risk components (use (4.9) and Lemma 5.1 below).

(MSA) Marginal VaR subadditivity:

$$VaR_\alpha[X_i + Y_i] \leq VaR_\alpha[X_i] + VaR_\alpha[Y_i], \quad i = 1, \ldots, d, \quad \forall \alpha \in (0,1). \quad (5.9)$$

Therefore, if (MSA) holds, then the marginal CTE components for Archimedean copulas are coherent risk measures. Similarly, if (MSA) holds, then the marginal VaR components for Archimedean copulas satisfy a subadditivity property similar to (5.8), namely

(SA) VaR Subadditivity:

$$VaR^i_\alpha[X+Y] \leq VaR^i_\alpha[X] + VaR^i_\alpha[Y], \quad i = 1, \ldots, d, \quad \forall \alpha \in (0,1). \quad (5.10)$$

This follows from the representation (4.8) and the fact $\varphi'(s) < 0$. In this situation, the marginal VaR components are also coherent risk measures. This fact holds because the properties (PH), (TI) and (MO) for the marginal VaR components are satisfied by Propositions 2.2 and 2.7 in Cousin and Di Bernardino (2013). Recall that there are many classes of distributions which satisfy the (MSA) property. In particular, linear elliptical portfolios, which are used in the Markowitz mean-variance portfolio theory, satisfy the (MSA) property, provided $\alpha \in [0.5, 1)$ (e.g. Embrechts *et al.*, 2002, Theorem 1). However, no attempt has been made so far to construct, if any, multivariate Archimedean risks that satisfy the (MSA) property.

Lemma 5.1. *For each completely monotone inverse generator φ^{-1}, function*

$$\gamma_d(s,\alpha) = \int_\alpha^s (-1)^{d-1}\frac{d}{dt}(\varphi^{-1})^{(d-1)}(\varphi(t)) \cdot \beta_d(s,t)dt \tag{5.11}$$

is non-negative on the interval $[\alpha, 1]$.

Proof. Since $\varphi'(s) < 0$, function $\beta_d(s,\alpha) = -\varphi'(s) \cdot (\varphi(\alpha) - \varphi(s))^{d-2}$ is non-negative on $[\alpha, 1]$. Similarly, the first term in the integrand is non-negative because φ^{-1} is completely monotone. The result is shown. □

In the univariate case, the CTE risk measure, which is a special instance of a distortion measure, not only satisfies the (MO) property but also preserves the more stringent stop-loss order (e.g. Hürlimann, 1998). It is interesting to investigate whether the following (IR) or stop-loss order preserving property holds for the marginal CTE components:

(IR) Increasing Risk: If $X_i \leq_{sl} Y_i$, $i = 1, \ldots, d$, then

$$CTE_\alpha^i[X] \leq CTE_\alpha^i[Y], \quad i = 1, \ldots, d, \quad \forall\alpha \in (0, 1). \tag{5.12}$$

To establish an (IR) property such as (5.12), it is often sufficient to derive it for the dangerousness order $\leq_D$, which is a special instance of the stop-loss order. Then, for a general proof, one uses the dangerousness characterization of the stop-loss order and invokes the dominated convergence theorem as in Hürlimann (1998, Theorem 2.2 and Section 3.2) (see also Hürlimann, 2005, proof of the theorem). Here we study the (IR) property for the class of Archimedean copulas. A simple sufficient condition for the validity of the (IR) property is the following one.

Lemma 5.2. *If function $\gamma_d(s,\alpha)$ is non-decreasing and non-negative in $s \in [\alpha, 1]$, then the (IR) property holds.*

Proof. Property (5.12) is shown as in Hürlimann (1998, Section 3.2). For fixed $i = 1, \ldots, d$, assume first that $X_i \leq_D Y_i$. Then one has $E[X_i] \leq E[Y_i]$ and there exists $q \in (0, 1)$ such that

$$VaR_s[X_i] \geq VaR_s[Y_i], \quad 0 \leq s < q, \quad VaR_s[X_i] \leq VaR_s[Y_i], \quad q \leq s \leq 1. \tag{5.13}$$

Using the representation (4.9), one must show that

$$I_d(\alpha) := \int_\alpha^1 \{VaR_s[Y_i] - VaR_s[X_i]\} \cdot \gamma_d(s,\alpha)ds \geq 0, \quad \forall\alpha \in [0, 1]. \tag{5.14}$$

If $\alpha \geq q$, then this is trivial by the second inequality in (5.13). Now, let $0 \leq \alpha < q < 1$. With the non-decreasing assumption and the fact that $\gamma_d(s,\alpha)$ is non-negative, one obtains

$$
\begin{aligned}
I_d(\alpha) = &- \int_{\alpha}^{q} \{VaR_s[X_i] - VaR_s[Y_i]\} \cdot \gamma_d(s,\alpha) ds \\
&+ \int_{q}^{1} \{VaR_s[Y_i] - VaR_s[X_i]\} \cdot \gamma_d(s,\alpha) ds \\
&\geq \gamma_d(q,\alpha) \cdot \int_{\alpha}^{1} \{VaR_s[Y_i] - VaR_s[X_i]\} ds \\
&\geq \gamma_d(q,\alpha) \cdot \int_{0}^{1} \{VaR_s[Y_i] - VaR_s[X_i]\} ds \\
&= \gamma_d(q,\alpha) \cdot \{E[Y_i] - E[X_i]\} \geq 0.
\end{aligned}
$$

Next, we assume that $X_i \leq_{sl} Y_i, i = 1, \ldots, d$. For each fixed $i = 1, \ldots, d$ there exists possibly an infinite sequence $Z_{i1}, Z_{i2}, \ldots, Z_{in}, \ldots$ such that $X_i = Z_{i1}, Z_{ik} \leq_D Z_{ik+1}$ and $Z_{ik} \to Y_i$ in the stop-loss convergence (convergence in distribution plus convergence in the mean). For $k = 1, 2, \ldots$, set $Z_1 = (Z_{11}, \ldots, Z_{d1}) = X, \ldots, Z_k = (Z_{1k}, \ldots, Z_{dk}), \ldots, Z_\infty = Y$, where components may be repeated if necessary. Now, since (4.9) is preserved under the dangerousness order, one has $CTE_\alpha^i[X] \leq CTE_\alpha^i[Z_n]$ for all $n \geq 1$. On the other hand, the relation $Z_{ik} \leq_D Z_{ik+1}$ implies $\min(Z_{ik}, x) \leq_D \min(Z_{ik+1}, x), \forall x$, hence $CTE_\alpha^i[\min(Z_n, x)] \leq CTE_\alpha^i[\min(Z_m, x)], \forall x, \forall m \geq n$. Using this, the result follows from the inequality

$$
\begin{aligned}
CTE_\alpha^i[Z_n] &= \lim_{x\to\infty} CTE_\alpha^i[\min(Z_n, x)] \\
&\leq \lim_{x\to\infty} \left\{ \lim_{m\to\infty} CTE_\alpha^i[[\min(Z_m, x)]] \right\} \\
&= \lim_{x\to\infty} CTE_\alpha^i[\min(Y, x)] = CTE_\alpha^i[Y].
\end{aligned}
$$

The first and third equalities follow through the continuity of representation (4.9). The second inequality is an application of the dominated convergence theorem for risks with finite support. □

By Lemma 5.1 the second assumption in Lemma 5.2 is always fulfilled for Archimedean copulas.

Corollary 5.1. (*Marginal (IR) property of bivariate CTE for Archimedean copulas*) *Let* $X = (X_1, X_2)$ *and* $Y = (Y_1, Y_2)$ *be bivariate risks that belong to an* *absolutely continuous bivariate Archimedean copula with generator φ. Then the (IR) property (5.12) is satisfied.*

Proof. If $d = 2$, one has from Examples 4.1 that $\gamma_2(s, \alpha) = 1 - \varphi'(\alpha)^{-1} \cdot \varphi'(s)$. Since $\varphi'(\alpha) < 0$ and $\varphi''(s) \geq 0$, the result follows from Lemma 5.2. □

In general, the (IR) property (5.12) can be investigated using the following formula.

Lemma 5.3. *For dimension $d \geq 3$, function $\gamma_d(s, \alpha)$ satisfies the recursion formula*

$$\begin{aligned}\gamma_d(s, \alpha) &= (d-2) \cdot \gamma_{d-1}(s, \alpha) \\ &\quad + (-1)^{d-1} (\varphi^{-1})^{(d-1)}(\varphi(\alpha)) \cdot \varphi'(s) \cdot (\varphi(\alpha) - \varphi(s))^{d-2}. \qquad (5.15)\end{aligned}$$

Proof. Applying a partial integration to Formula (4.9), one obtains

$$\begin{aligned}\gamma_d(s, \alpha) &= \int_\alpha^s (-1)^{d-1} \frac{d}{dt} (\varphi^{-1})^{(d-1)}(\varphi(t)) \cdot \beta_d(s, t) dt \\ &= (-1)^d \varphi'(s) (\varphi^{-1})^{(d-1)}(\varphi(t)) \cdot \varphi'(s) \cdot (\varphi(t) - \varphi(s))^{d-2} \Big|_{t=\alpha}^{t=s} \\ &\quad + (d-2)(-1)^{d-1} \cdot \int_\alpha^s (\varphi^{-1})^{(d-1)}(\varphi(t)) \cdot \varphi'(t) \cdot \varphi'(s) \cdot (\varphi(t) - \varphi(s))^{d-3} dt \\ &= (d-2)(-1)^{d-2} \cdot \int_\alpha^s \frac{d}{dt} (\varphi^{-1})^{(d-2)}(\varphi(t)) \cdot (-\varphi'(s)) \cdot (\varphi(t) - \varphi(s))^{d-3} dt \\ &\quad + (-1)^{d-1} (\varphi^{-1})^{(d-1)}(\varphi(\alpha)) \cdot \varphi'(s) \cdot (\varphi(\alpha) - \varphi(s))^{d-2},\end{aligned}$$

which coincides with (5.15) by the definition of $\gamma_{d-1}(s, \alpha)$. □

Unfortunately, for dimension $d \geq 3$, a general proof of the (IR) property cannot be given on the basis of Lemma 5.3, as demonstrated by the following example.

Example 5.2. *The (IR) property for the trivariate Clayton copula*
For $d = 3$, consider Recursion (5.15), that is

$$\gamma_3(s, \alpha) = \gamma_2(s, \alpha) - \frac{\varphi''(\alpha)}{\varphi'(\alpha)^3} \cdot \varphi'(s) \cdot (\varphi(\alpha) - \varphi(s))^{d-2}.$$

Since $\gamma_2(s, \alpha) = 1 - \varphi'(\alpha)^{-1} \cdot \varphi'(s)$, the derivativewith respect to $s \in [\alpha, 1]$ can be recast into the form

$$\frac{d}{ds}\gamma_3(s,\alpha) = -\frac{\varphi''(\alpha)}{\varphi'(\alpha)} \cdot \left\{ \frac{\varphi''(s)}{\varphi''(\alpha)} - \left(\frac{\varphi'(s)}{\varphi'(\alpha)} \right)^2 \right\}$$
$$- \frac{\varphi''(\alpha)}{\varphi'(\alpha)^3} \cdot \varphi''(s) \cdot (\varphi(\alpha) - \varphi(s)). \tag{5.16}$$

Obviously, since $\varphi'(\alpha) < 0$ and $\varphi''(s) \geq 0$, expression (5.16) will be non-negative, provided the curly bracket is there. Now for the Clayton copula with generator $\varphi(s) = \theta^{-1} \cdot (s^{-\theta} - 1), \theta \in (-1, \infty)$, one sees immediately that this bracket is non-negative over the interval $s \in [\alpha, 1]$, provided $\theta \geq 0$, and the (IR) property holds in this situation. On the other hand, a calculation shows that derivative (5.16) is negative for $\theta < 0$, and the sufficient criterion from Lemma 5.2 is not conclusive in this situation. In fact, it is even possible to construct herewith a counterexample to the (IR) property. Let $X = (X_1, X_2, X_3)$ and $Y = (Y_1, Y_2, Y_3)$ be trivariate risks with marginal exponential components $X_i \sim Exp(\mu)$ and marginal shifted Pareto components $Y_i \sim Par(\mu(\lambda - 1), \lambda)$. The VaR measures are given by

$$VaR_s[X_i] = -\mu \cdot \ln(1-s),\ VaR_s[Y_i] = \mu(\lambda-1)(1-s)^{-1/\lambda}, \quad s \in (0,1).$$

Clearly, the means are equal and one has $X_i \leq_{sl} Y_i, i = 1, .2, 3$ (e.g. Hesselager, 1995, Table 2). A numerical integration with parameters $\theta = -0.75, \alpha = 0.95, \lambda = 1.1, \mu = 1,000$ shows that

$$I_3(\alpha) := \int_\alpha^1 \{VaR_s[Y_i] - VaR_s[X_i]\} \cdot \gamma_3(s,\alpha)ds = -0.10805 < 0,$$

which contradicts the (IR) property (5.14).

6. Archimedean risk conclusions and outlook

To conclude, it might be helpful to summarize what has been accomplished. Starting point are the recent multivariate VaR and CTE risk measures introduced by Cousin and Di Bernardino (2013, 2014). Based on a short review of some main general properties in Theorem 5.1, we ask for conditions under which the marginal VaR and CTE components of multivariate risks define coherent risk measures. Also, a main emphasis has been put on the study of additional properties of these risk measures under a fixed Archimedean copula structure. Several results are worthwhile to be mentioned.

- The marginal CTE components of multivariate risks with independent components are coherent risk measures in the sense of Artzner *et al.* (1999) (all the required properties (PH), (TI), (MO) and (SA) are already derived in Cousin and Di Bernardino, 2014).

- Given a fixed Archimedean copula, the marginal VaR and CTE components are integral transforms of the corresponding univariate VaR measures. Formulas (4.8) and (4.9) are essential ingredients in the derivation of several results (proof of Corollary 4.1, Examples 4.1, coherent risk measures under the (MSA) property and Lemma 5.2 used to show Corollary 5.1).
- The marginal CTE components of multivariate Archimedean risks satisfy besides (PH), (TI) and (MO) the properties of (SL) and (VIA) (Theorem 5.2).
- Under a fixed Archimedean copula, the marginal VaR and CTE components of multivariate risks are coherent risk measures under the (MSA) property. An open question is to construct, if any, multivariate Archimedean distributions that satisfy the (MSA) property.
- The marginal (IR) property of the bivariate CTE under a fixed Archimedean copula holds (Corollary 5.1). A counterexample to the (IR) property for the trivariate Clayton copula has been constructed in Example 5.2.

Finally, some further comments and a brief overview on related developments should be given. The (MO) (respectively (IR)) preserving property tells us that the CTE components are consistent with ordering of risks in the sense that profit-seeking (risk-averse) decision makers require higher CTE by increasing risk in stochastic order (stop-loss order). In the univariate case, the (IR) property is a part of Hürlimann (2001, Theorem 1.1), which provides a characterization of stop-loss order by CTE functional (see also Shaked and Shanthikumar, 2007, condition (4.A.8) and p. 228). The stop-loss (IR) property is *weaker* than the (MO) property required in the classical definition by Artzner *et al.* (1999). It is not difficult to construct examples of coherent risk measures in the classical sense that do not satisfy the stop-loss increasing property (e.g. De Giorgi, 2005). The weaker axiomatic risk measurement system (PH), (TI), (IR), (SA) and (SL) finds applications in insurance pricing (e.g. Wang *et al.*, 1997; Hürlimann, 1998, 2002a; Young, 2004), economic capital modelling (e.g. Wirch and Hardy, 1999; Hürlimann, 2004b; Goovaerts *et al.*, 2012) and the portfolio theory (e.g. Hürlimann, 2002b; De Giorgi, 2005; Adam *et al.*, 2008; Roman and Mitra, 2009; Sereda *et al.*, 2009; Feng, 2011; Feng and Tan, 2012). Finally, it is also worthwhile to mention that if the (PH) property holds, then (SA) is equivalent with the axiom for a *convex risk measure* (e.g. Föllmer and Schied, 2002, 2010):

(CX) Convexity: $\forall 0 \leq \lambda \leq 1$

$$\begin{aligned} CTE_{\alpha}^{i}\left[\lambda X+(1-\lambda) Y\right] &\leq \lambda \cdot CTE_{\alpha}^{i}\left[X\right] \\ &+(1-\lambda) \cdot CTE_{\alpha}^{i}\left[Y\right], i=1, \ldots, d, \quad \forall \alpha \in (0,1) \end{aligned} \tag{6.1}$$

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